数理解析研究所講究録
1519 巻 2006 年 137-145

REPRESENTATION PROPERTY OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON THE UPPER HALF-SPACES

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1. INTRODUCTION

Let $\mathbf{H}$ denote the upper half space $\mathbf{R}^{n-1} \times \mathbf{R}_+$ where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will write points $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \le p < \infty$, let $b^p_\alpha = b^p_\alpha(\mathbf{H})$ denote the weighted harmonic Bergman space consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$\|u\|_{L^p_\alpha} := \left(\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z)\right)^{1/p} < \infty$$

where $dV_\alpha(z) = z_n^\alpha dz$ and dz is the Lebesque measure on $\mathbf{R}^n$. Then we can see easily that the space b^p_α is a Banach space. In particular, b^2_α is a Hilbert space. Hence, there is a unique Hilbert space orthogonal projection Π_α of L^2_α onto b^2_α which is called the weighted harmonic Bergman projection. It is known that this weighted harmonic Bergman projection can be realized as an integral operator against the weighted harmonic Bergman kernel $R_\alpha(z, w)$. See section 2.

The purpose of this paper is to survey [8] concerning the representation property of b^p_α-functions and the interpolation by b^p_α-functions.

In the holomorphic case representation and interpolation properties of Berg man functions have been studied in [5] and [11]. In [5], the representation properties of harmonic Bergman functions, as well as harmonic Bloch functions, were also proved on the unit ball in $\mathbf{R}^n$. See [2] for the interpolation properties of holomorphic (little) Bloch functions. On the setting of the half-space of $\mathbf{R}^n$, Choe and Yi [6] have studied these two properties of harmonic Bergman spaces. In [6], the harmonic (little) Bloch spaces are also considered as limiting spaces of b^p.

2. PRELIMINARIES

First, we introduce the fractional derivative. Let D denote the differentiation with respect to the last component and let $u \in b^p_\alpha$. Then the mean value

This research was supported by KOSEF(R01-2003-000-10243-0) .

property, Jensen's inequality and Cauchy's estimate yield

$$|D^k u(z)| \le c z_n^{-(n+\alpha)/p-k} \tag{2.1}$$

for each $z \in \mathbf{H}$ and for every nonnegative integer k.

Let $\mathcal{F}_\beta$ be the collection of all functions v on $\mathbf{H}$ satisfying $|v(z)| \le c z_n^{-\beta}$ for $\beta > 0$ and let $\mathcal{F} = \cup_{\beta>0}\mathcal{F}_\beta$. If $v \in \mathcal{F}$, then $v \in \mathcal{F}_\beta$ for some $\beta > 0$. In this case, we define the fractional derivative of v of order $-s$ by

$$\mathcal{D}^{-s} v(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} v(z', z_n + t)\, dt \tag{2.2}$$

for the range $0 < s < \beta$. (Here, Γ is the Gamma function.)

If $u \in b^p_\alpha$, then for every nonnegative integer k, $D^k u \in \mathcal{F}$ by (2.1). Thus for $s > 0$, we define the fractional derivative of u of order s by

$$\mathcal{D}^s u = \mathcal{D}^{-([s]-s)} D^{[s]} u. \tag{2.3}$$

Here, $[s]$ is the smallest integer greater than or equal to s and $\mathcal{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < [s] - s - 1 < 0$ and $[s] \ge 1$. Thus we know from (2.1) that, for each $z \in \mathbf{H}$ and for every $u \in b^p_\alpha$, the integral

$$\mathcal{D}^s u(z) = \frac{1}{\Gamma([s]-s)} \int_0^\infty t^{[s]-s-1} D^{[s]} u(z', z_n + t)\, dt$$

always makes sense.

Let $P(z,w)$ be the extended Poisson kernel on $\mathbf{H}$ and put $P_z = P(z,\cdot)$. More explicitly,

$$P_z(w) = P(z,w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \overline{w}|^n}$$

where $z, w \in \mathbf{H}$ and $\overline{w} = (w', -w_n)$ and B is the open unit ball in $\mathbf{R}^n$. It is known that the weighted harmonic Bergman projection Π_α of L^2_α onto b^2_α is given by

$$\Pi_\alpha f(z) = \int_{\mathbf{H}} f(w) R_\alpha(z,w)\, dV_\alpha(w)$$

for all $f \in L^2_\alpha$. Here $R_\alpha(z,w)$ denotes the weighted harmonic Bergman kernel whose explicit formula is given by

$$R_\alpha(z,w) = C_\alpha \mathcal{D}^{\alpha+1} P_z(w) \tag{2.4}$$

where $C_\alpha = (-1)^{[\alpha]+1} 2^{\alpha+1} / \Gamma(\alpha+1)$. Also, it is known that

$$|\mathcal{D}^\beta_{z_n} R_\alpha(z,w)| \le \frac{C}{|z - \overline{w}|^{n+\alpha+\beta}} \tag{2.5}$$

for all $z, w \in \mathbf{H}$. Here, $\beta > -n - \alpha$ and the constant C is dependent only on n, α and β. Using (2.5), we know $R_\alpha(z,\cdot) \in b^q_\alpha$ for all $1 < q \le \infty$. Thus, Π_α

is well defined whenever $f \in L^p_\alpha$ for $1 \le p < \infty$. Also, for $1 \le p < \infty$, $u \in b^p_\alpha$, $z \in \mathbf{H}$, we have the reproducing formula

$$u(z) = \int_{\mathbf{H}} u(w) R_\beta(z,w)\, dV_\beta(w) \tag{2.6}$$

whenever $\beta \ge \alpha$. Furthermore, we have a useful norm equivalence. If $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$, then

$$\|u\|_{L^p_\alpha} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_{L^p_\alpha} \tag{2.7}$$

as u ranges over b^p_α.

Set $z_0 = (0,1)$. A harmonic function u on $\mathbf{H}$ is called a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup_{w \in \mathbf{H}} w_n |\nabla u(w)| < \infty,$$

where ∇u denotes the gradient of u. We let $\mathcal{B}$ denote the set of Bloch functions on $\mathbf{H}$ and let $\widetilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at z_0. Then the space $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\|\ \|_{\mathcal{B}}$.

A function $u \in \widetilde{\mathcal{B}}$ is called a harmonic little Bloch function if it has the following vanishing condition

$$\lim_{z \to \partial^\infty \mathbf{H}} z_n |\nabla u(z)| = 0$$

where $\partial^\infty \mathbf{H}$ denotes the union of $\partial \mathbf{H}$ and $\{\infty\}$. Let $\widetilde{\mathcal{B}}_0$ denote the set of all harmonic little Bloch functions on $\mathbf{H}$. It is not hard to verify that $\widetilde{\mathcal{B}}_0$ is a closed subspace of $\widetilde{\mathcal{B}}$. Let $\mathcal{C}_0$ denote the set of all continuous functions on $\mathbf{H}$ vanishing at ∞.

Because $R_\alpha(z,\cdot)$ is not in L^1_α, $\Pi_\alpha f$ is not well defined for $f \in L^\infty$. So we need the following modified Bergman kernel. For $z, w \in \mathbf{H}$, define

$$\widetilde{R}_\alpha(z,w) = R_\alpha(z,w) - R_\alpha(z_0,w).$$

Then, there is a constant $C = C(n,\alpha)$ such that

$$|\widetilde{R}_\alpha(z,w)| \le C \left(\frac{|z-z_0|}{|z-\overline{w}|^{n+\alpha}|z_0-\overline{w}|} + \frac{|z-z_0|}{|z-\overline{w}||z_0-\overline{w}|^{n+\alpha}} \right) \tag{2.8}$$

for all $z, w \in \mathbf{H}$. Thus, (2.8) implies that $\widetilde{R}_\alpha(z,\cdot) \in L^1_\alpha$ for each fixed $z \in \mathbf{H}$ and thus we can define $\widetilde{\Pi}_\alpha$ on L^∞ by

$$\widetilde{\Pi}_\alpha f(z) = \int_{\mathbf{H}} f(w) \widetilde{R}_\alpha(z,w)\, dV_\alpha(w)$$

for $f \in L^\infty$. It turns out that $\widetilde{\Pi}_\alpha$ is a bounded linear map from L^∞ onto $\widetilde{\mathcal{B}}$. Also, $\widetilde{\Pi}_\alpha$ has the following property: If $\gamma > 0$ and $v \in \widetilde{\mathcal{B}}$ then

$$\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z) = Cv(z) \tag{2.9}$$

where $C = C(\alpha, \gamma)$. The Bloch norm is also equivalent to the normal derivative norm : If $\gamma > 0$, then

$$\|u\|_{\mathcal{B}} \approx \|w_n^{\gamma} \mathcal{D}^{\gamma} u\|_{\infty} \tag{2.10}$$

as u ranges over $\widetilde{\mathcal{B}}$. (See [7] for details.)

3. Technical Lemmas

We first introduce a distance function on $\mathbf{H}$ which is useful for our purposes. The pseudohyperbolic distance between $z, w \in \mathbf{H}$ is defined by

$$\rho(z, w) = \frac{|z - w|}{|z - \overline{w}|}.$$

This ρ is an actual distance. (See [6].) Note that ρ is horizontal translation invariant and dilation invariant. In particular,

$$\rho(z, w) = \rho(\phi_a(z), \phi_a(w)) \tag{3.1}$$

for $z, w \in \mathbf{H}$ where $\phi_a (a \in \mathbf{H})$ denotes the function defined by

$$\phi_a(z) = \left(\frac{z' - a'}{a_n}, \frac{z_n}{a_n} \right)$$

for $z \in \mathbf{H}$. Note that the Jacobian of ϕ_a^{-1} is a_n^n. For $z \in \mathbf{H}$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z with radius δ. Note that $\phi_z(E_\delta(z)) = E_\delta(z_0)$ by the invariance property (3.1). Also, simple calculation shows that

$$E_\delta(z) = B\left(\left(z', \frac{1 + \delta^2}{1 - \delta^2} z_n \right), \frac{2\delta}{1 - \delta^2} z_n \right) \tag{3.2}$$

so that $B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1 - \delta)^{-1} z_n)$ where $B(z, r)$ denotes the Euclidean ball centered at z with radius r. From (3.2), we have two lemmas. For proofs of the following lemmas, see [6].

Lemma 3.1. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{z_n}{w_n} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

This lemma implies the following lemma.

Lemma 3.2. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{|z - \overline{s}|}{|w - \overline{s}|} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$$

for all $s \in \mathbf{H}$.

The following lemma is used to prove the representation theorem. If α is a nonnegative integer, then it is proved in [6].

Lemma 3.3. *Let $\alpha > -1$ and β be real. Then*

$$\left|z_n^\beta R_\alpha(s,z) - w_n^\beta R_\alpha(s,w)\right| \le C\rho(z,w)\frac{z_n^\beta}{|z-\overline{s}|^{n+\alpha}}$$

whenever $\rho(z,w) < 1/2$ and $s \in \mathbf{H}$.

Let $\alpha > -1$ and let $1 \le p < \infty$. Define Π_β on the weighted Lebesque space L^p_α by

$$\Pi_\beta f(z) = \int_{\mathbf{H}} f(w) R_\beta(z,w)\, dV_\beta(w)$$

for $f \in L^p_\alpha$ and $z \in \mathbf{H}$. Then we have the following two lemmas from [7].

Lemma 3.4. *Suppose $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta+1)p$. Then Π_β is bounded projection of L^p_α onto b^p_α.*

Lemma 3.5. *For $b < 0, -1 < a + b$, there exists a constant $C = C(a,b)$ such that*

$$\int_{\mathbf{H}} \frac{w_n^{a+b}}{|z-\overline{w}|^{n+a}}\, dw \le C z_n^b$$

for every $z, w \in \mathbf{H}$.

Lemma 3.6. *Let $\alpha > -1$, $1 \le p < \infty$ and let $(1+\alpha)/p + \gamma > 0$. Suppose $0 < \delta < 1$. Then*

$$z_n^{n+p\gamma} |\mathcal{D}^\gamma u(z)|^p \le \frac{C}{\delta^{n+pk}} \int_{E_\delta(z)} |u(w)|^p\, dw$$

for all $z \in \mathbf{H}$ and for every u harmonic on $\mathbf{H}$ where $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \le -1$. The constant $C = C(n,p,\gamma)$ is independent of δ.

If γ satisfies the condition of Lemma 3.6, we can show $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$. If γ is a nonnegative integer, then $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$, because it is a partial derivative of a harmonic function. If γ is not a nonnegative integer, we see also $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$ by passing the Laplacian through the integral.

The notation $|E|$ denotes the Lebesque measure of a Borel subset E of $\mathbf{H}$. Let $|E|_\alpha$ denote $V_\alpha(E)$. The following lemma is proved by using the mean value property and Cauchy's estimates. The notation $d(E,F)$ denotes the euclidean distance between two sets E and F.

Lemma 3.7. *Suppose u is harmonic on some proper open subset Ω of $\mathbf{R}^n$. Let $\alpha > -1$ and let $1 \le p < \infty$. Then, for a given open ball $E \subset \Omega$,*

$$\int_E |u(z) - u(a)|^p\, dV_\alpha(z) \le C \frac{|E|^{p/n}|E|_\alpha}{d(E,\partial\Omega)^{n+p}} \int_\Omega |u(w)|^p\, dw$$

for all $a \in E$. The constant C depends only on n, α and p.

4. Representation theory

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $0 < \delta < 1$. We say that $\{z_m\}$ is δ-separated if the balls $E_\delta(z_m)$ are pairwise disjoint or simply say that $\{z_m\}$ is separated if it is δ-separated for some δ. Also, we say that $\{z_m\}$ is a δ-lattice if it is $\delta/2$-separated and $\mathbf{H} = \bigcup E_\delta(z_m)$. Note that any "maximal" $\delta/2$-separated sequence is a δ-lattice.

From [4] and [6], we have the following three lemmas.

Lemma 4.1. *Fix a $1/2$-lattice $\{a_m\}$ and let $0 < \delta < 1/8$. If $\{z_m\}$ is a δ-lattice, then we can find a rearrangement $\{z_{ij} : i = 1, 2, \ldots, j = 1, 2, \ldots, N_i\}$ of $\{z_m\}$ and a pairwise disjoint covering $\{D_{ij}\}$ of $\mathbf{H}$ with the following properties:*
(a) $E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij})$
(b) $E_{1/4}(a_i) \subset \cup_{j=1}^{N_i} D_{ij} \subset E_{5/8}(a_i)$
(c) $z_{ij} \in E_{1/2}(a_i)$
for all $i = 1, 2, \ldots,$ and $j = 1, 2, \ldots, N_i$.

Lemma 4.2. *Let $r > 0$ and let $0 < r\eta < 1$. If $\{z_m\}$ is an η-separated sequence, then there is a constant $M = M(n, r, \eta)$ such that more than M of the balls $E_{r\eta}(z_m)$ contain no point in common.*

Lemma 4.3. *Let N_i be the sequence defined in Lemma 4.1. Then*

$$\sup_i N_i \leq C\delta^{-n}$$

for some constant C depending only on n.

Analysis similar to that for the proof of Lemma 3.4 shows the following lemma which will be used in the proof of Proposition 4.5.

Lemma 4.4. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $f \in L^p_\alpha$, define*

$$\Phi_\beta f(z) = \int_{\mathbf{H}} f(w) \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta}} \, dw$$

for $z \in \mathbf{H}$. Then, $\Phi_\beta : L^p_\alpha \to L^p_\alpha$ is bounded.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $(\lambda_m) \in l^p$, let $Q_\beta(\lambda_m)$ denote the series defined by

$$Q_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p} R_\beta(z, z_m) \tag{4.1}$$

for $z \in \mathbf{H}$. For a sequence $\{z_m\}$ good enough, $Q_\beta(\lambda_m)$ will be harmonic on $\mathbf{H}$. We say that $\{z_m\}$ is a b^p_α-representing sequence of order β if $Q_\beta(l^p) = b^p_\alpha$. Lemma 4.4 implies the following proposition which shows $Q_\beta(l^p) \subset b^p_\alpha$ if the underlying sequence is separated.

Proposition 4.5. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Suppose $\{z_m\}$ is a δ-separated sequence. Then $Q_\beta : l^p \to b^p_\alpha$ is bounded.*

The following theorem is the b^p_α-representation result under the lattice density condition.

Theorem 4.6. *Let $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta+1)p$. Then there exists $\delta_0 > 0$ with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $Q_\beta : l^p \to b^p_\alpha$ be the associated linear operator as in (4.1). Then there is a bounded linear operator $\mathcal{P}_\beta : b^p_\alpha \to l^p$ such that $Q_\beta \mathcal{P}_\beta$ is the identity on b^p_α. In particular, $\{z_m\}$ is a b^p_α-representing sequence of order β.*

Since $\mathcal{D}^\gamma u$ is harmonic and we have (2.7), we can have similar result with Proposition 4.8 of [6].

Proposition 4.7. *Let $\alpha > -1$, $1 \le p < \infty$ and let $(1+\alpha)/p + \gamma > 0$. If $\{z_m\}$ is a δ-lattice with δ sufficiently small, then*

$$\|u\|^p_{L^p_\alpha} \approx \sum z_{mn}^{n+\alpha+p\gamma} |\mathcal{D}^\gamma u(z_m)|^p$$

as u ranges over b^p_α.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $\beta > -1$. For $(\lambda_m) \in l^\infty$, let

$$\widetilde{Q}_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{n+\beta} \widetilde{R}_\beta(z, z_m) \tag{4.2}$$

for $z \in \mathbf{H}$. We say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}$-representing sequence of order β if $\widetilde{Q}_\beta(l^\infty) = \widetilde{\mathcal{B}}$. We also say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}_0$-representing sequence of order β if $\widetilde{Q}_\beta(\mathcal{C}_0) = \widetilde{\mathcal{B}}_0$. Then we have the result which shows that a separated sequence represents a part of the whole space.

Proposition 4.8. *Let $\beta > -1$ and suppose $\{z_m\}$ is a δ-separated sequence. Then, $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ is bounded. In addition, $\widetilde{Q}_\beta$ maps $\mathcal{C}_0$ into $\widetilde{\mathcal{B}}_0$.*

If γ is a positive integer, then the following lemma is proved in [6].

Lemma 4.9. *Let $\gamma > 0$. Then*

$$|z_n^\gamma \mathcal{D}^\gamma u(z) - w_n^\gamma \mathcal{D}^\gamma u(w)| \le C\rho(z,w)\|u\|_{\mathcal{B}}$$

for all $z, w \in \mathbf{H}$ and $u \in \widetilde{\mathcal{B}}$.

The following theorem is the limiting version of the b^p_α-representation theorem.

Theorem 4.10. *Let $\beta > -1$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ be the associated linear operator as in (4.2). Then there exists a bounded linear operator $\widetilde{\mathcal{P}}_\beta : \widetilde{\mathcal{B}} \to l^\infty$ such that $\widetilde{Q}_\beta \widetilde{\mathcal{P}}_\beta$ is the identity on $\widetilde{\mathcal{B}}$. Moreover, $\widetilde{\mathcal{P}}_\beta$ maps $\widetilde{\mathcal{B}}_0$ into $\mathcal{C}_0$. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-representing and $\widetilde{\mathcal{B}}_0$-representing sequence of order β.*

Lemma 4.9 yields the following result for $\widetilde{\mathcal{B}}$ analogous to Proposition 4.7.

Proposition 4.11. *Let $\gamma > 0$. Let $\{z_m\}$ be a δ-lattice with δ sufficiently small. Then*

$$\|u\|_{\mathcal{B}} \approx \sup_m z_{mn}^{\gamma} |\mathcal{D}^{\gamma} u(z_m)|$$

as u ranges over $\widetilde{\mathcal{B}}$.

5. Interpolation theory

Let $\{z_m\}$ be a sequence on $\mathbf{H}$. Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. For $u \in b^p_\alpha$, let $T_\gamma u$ denote the sequence of complex numbers defined by

$$T_\gamma u = \left(z_{mn}^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(z_m)\right). \tag{5.1}$$

If $T_\gamma(b^p_\alpha) = l^p$, we say that $\{z_m\}$ is a b^p_α-interpolating sequence of order γ.

The following two lemmas are used to prove that separation is necessary for b^p_α-interpolation.

Lemma 5.1. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Let $\{z_m\}$ be a b^p_α-interpolating sequence of order γ. Then $T_\gamma : b^p_\alpha \to l^p$ is bounded.*

The following lemma is a b^p_α-version of Lemma 4.9 concerning $\widetilde{\mathcal{B}}$-functions. If γ is a nonnegative integer, then the following lemma is proved in [6].

Lemma 5.2. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Then,*

$$\left| z_n^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(z) - w_n^{(n+\alpha)/p+\gamma} \mathcal{D}^{\gamma} u(w) \right| \le C\rho(z,w) \|u\|_{L^p_\alpha}$$

for all $z, w \in \mathbf{H}$ and $u \in b^p_\alpha$.

Proposition 5.3. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Every b^p_α-interpolating sequence of order γ is separated.*

For interpolation, we need the sufficient separation condition.

Theorem 5.4. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p+\gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $T_\gamma : b^p_\alpha \to l^p$ be the associated linear operator as in (5.1). Then there is a bounded linear operator $S_\gamma : l^p \to b^p_\alpha$ such that $T_\gamma S_\gamma$ is the identity on l^p. In particular, $\{z_m\}$ is a b^p_α-interpolating sequence of order γ.*

Let $\gamma > 0$ and let $\{z_m\}$ be a sequence in $\mathbf{H}$. For $u \in \widetilde{\mathcal{B}}$, define

$$\widetilde{T}_\gamma u = \left(z_{mn}^{\gamma} \mathcal{D}^{\gamma} u(z_m)\right). \tag{5.2}$$

Then (2.10) implies the operator

$$\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \to l^\infty$$

is bounded. If $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}) = l^\infty$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}$-interpolating sequence of order γ. Also, if $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}_0) = \mathcal{C}_0$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.

The following proposition shows that separation is also necessary for $\widetilde{\mathcal{B}}_0$ interpolation. Since we have Lemma 4.9, the proof of the following proposition is the same as that of Proposition 5.6 in [6].

Proposition 5.5. *Let $\gamma > 0$. Every $\widetilde{\mathcal{B}}$-interpolating sequence of order γ is separated. Also, every $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ is separated.*

Theorem 5.6. *Let $\gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \to l^\infty$ be the associated linear operator as in (5.2). Then there exists a bounded linear operator $\widetilde{S}_\gamma : l^\infty \to \widetilde{\mathcal{B}}$ such that $\widetilde{T}_\gamma \widetilde{S}_\gamma$ is the identity on l^∞. Moreover, $\widetilde{S}_\gamma$ maps C_0 into $\widetilde{\mathcal{B}}_0$. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-interpolating and $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.*

References

[1] E. Amar, *Suites d'interpolation pour les classes de Bergman de la boule du polydisque de $\mathbb{C}^n$*, Canadian J. Math. **30** (1978), 711–737.

[2] K. R. M. Attle, *Interpolating sequences for the derivatives of Bloch functions*, Glasgow Math. J. **34** (1992), 35-41.

[3] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer-Verlag, New York 1992.

[4] B. R. Choe, H. Koo and H. Yi, *Positive Toeplitz operators between the harmonic Bergman spaces*, Potential Analysis **17** (2002), 307–335.

[5] R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque **77** (1980), 11–66.

[6] B. R. Choe and H. Yi, *Representations and interpolations of harmonic Bergman functions on half-spaces*, Nagoya Math.J. **151** (1998), 51–89.

[7] H. Koo, K. Nam, and H. Yi, *Weighted harmonic Bergman functions on half-spaces* J. Korean Math. Soc. **42** (2005), no. 5, 975–1002.

[8] K. Nam, *Representations and interpolations of weighted harmonic Bergman functions,* Rocky Mountain J. Math. **36** (2006), no. 1, 237–263.

[9] F. Ricci and M. Taibleson, *Boundary values of harmonic functions in mixed norm spaces and their atomic structure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), no. 1, 1–54.

[10] F. Ricci and M. Taibleson, *Representation theorems for harmonic functions in mixed norm spaces on the half plane*, Rend. Circ. Mat. Palermo (2) (1981), suppl. 1, 121–127.

[11] R. Rochberg, *Interpolation by functions in Bergman spaces*, Michigan Math. J. **29** (1982), 229–236.

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